JOURNAL OF APPROXIMATION THEORY 49, 31–40 (1987)

A Geometric Algorithm for Approximating Semicontinuous Function*

GERALD BEER

Department of Mathematics, California State University, Los Angeles, California 90032, U.S.A.

Communicated by E. W. Cheney

Received June 29, 1984

1. INTRODUCTION

A real valued function h defined on a topological space X is called *upper semicontinuous* (u.s.c.) if any of the following equivalent conditions is satisfied:

(1) for each $\alpha \in R$, $h^{-1}((-\infty, \alpha))$ is open in X;

(2) for each x in X and $\varepsilon > 0$ there exists a neighborhood $V(x, \varepsilon)$ of x such that $h(z) < h(x) + \varepsilon$, provided $z \in V(x, \varepsilon)$;

(3) the *hypograph* of h, $\{(x, \alpha): \alpha \leqslant h(x)\}$, is closed in $X \times R$.

We call $h: X \to R$ *lower semicontinuous* (l.s.c.) if $-h$ is u.s.c. Those topological spaces for which each upper semicontinuous function is the pointwise limit of a decreasing sequence of continuous functions are the *perfectly normal spaces* [10], i.e., spaces in which each closed subset is a G_δ set. That such approximations exist for metric spaces was first proved by Hahn [3]. It is the purpose of this article to set forth in the context of metric spaces a natural geometric algorithm that yields such a sequence of continuous functions, beginning with any continuous function f majorizing our u.s.c. function. If f is Lipschitz, then the algorithm will generate Lipschitz functions.

Before describing the algorithm, we recall that if $h: X \to R$ is arbitrary, then the *upper envelope* h^* of h is defined as

$$h^*(x) = \sup\{\limsup_{n \to \infty} h(x_n): \lim_{n \to \infty} x_n = x\}.$$

* 1980 Mathematics Subject Classification: Primary: 26A15, 65S05; Secondary: 40A30, 41A25.

0021-9045/87 $3.00

640/49/1-3

Alternatively, h^* is that function whose hypograph is the closure of the hypograph of h [6]. Thus, h^* is the smallest extended real valued u.s.c. function that majorizes h. We shall also need a metric for $X \times R$. If d is the metric for X, we employ the *box metric*:

$$\rho[(x_1, \alpha_1), (x_2, \alpha_2)] = \max\{d(x_1, x_2), |\alpha_2 - \alpha_1|\}.$$

2. The Algorithm

LEMMA 1. *Let $\langle X, d\rangle$ be a metric space. Let $h: X \to R$ be arbitrary, and suppose $f \in C(X, R)$ majorizes h. Define $\varphi(f, h): X \to R$* by

$$\varphi(f, h)(x) = \inf\{\rho[(x, f(x)), (y, h(y))]: y \in X\}.$$

Then $f - \varphi(f, h)$ is continuous and majorizes h^*, the upper envelope of h. If f is Lipschitz, then so is $f - \varphi(f, h)$.

Proof. First, notice that $\varphi(f, h)(x)$ just gives the distance from $(x, f(x))$ to the graph of h. Since $(x, \alpha) \to$ the distance of (x, α) from the graph of h is a Lipschitz function on $X \times R$ with Lipschitz constant one and $x \to (x, f(x))$ is continuous, their composition $\varphi(f, h)$ is continuous. Suppose now that f is Lipschitz with constant $K \geq 1$. We then have $\rho[(x, f(x)), (w, f(w))] \leq Kd(x, w)$ for each x and w in X. Since the difference between the distances of any two points in a metric space to a given set is less than or equal to the distance between the two points, we get

$$|\varphi(f, h)(x) - \varphi(f, h)(w)| \leq Kd(x, w),$$

whence $\varphi(f, h)$ is Lipschitz. Thus, $f - \varphi(f, h)$ is Lipschitz (with constant $2K$).

To show that $f - \varphi(f, h)$ majorizes h^*, fix x in X and choose $\{x_n\}$ convergent to x for which $\lim_{n \to \infty} h(x_n) = h^*(x)$. Since f is u.s.c. and $f \geq h$, we have $f(x) - h^*(x) \geq 0$ and

$$\begin{aligned} f(x) - h^*(x) &= \rho[(x, f(x)), (x, h^*(x))] \\ &= \sup_{k \in Z^+} \inf\{\rho[(x, f(x)), (x_n, h(x_n))]: n \geq k\} \\ &\geq \inf\{\rho[(x, f(x)), (x_n, h(x_n))]: n \in Z^+\} \\ &\geq \inf\{\rho[(x, f(x)), (y, h(y))]: y \in X\} \\ &= \varphi(f, h)(x). \end{aligned}$$

We remark that $\varphi(f, h)$ can be the zero function even if for all x,

$f(x) > h(x)$. For example, let $X = [0, 1]$, let f be the zero function on X, and let $h\colon X \to [-1, 0)$ be given by

$$h(x) = -\frac{1}{2^n}, \qquad \text{if } x = \frac{q}{2^n},\ q \text{ an odd integer},$$
$$= -1, \qquad \text{otherwise.}$$

However, if h is u.s.c. at x and $h(x) < f(x)$, then $\varphi(f, h)(x) > 0$; otherwise, for some sequence $\{x_n\}$ convergent to x, we would have $\lim_{n \to \infty} h(x_n) = f(x) > h(x)$.

If we iterate the procedure described above, we produce a decreasing sequence of functions convergent to h^*. In the sequel we shall use the notation $\varphi(f, h)$ described in Lemma 1 freely.

THEOREM 1. *Let $\langle X, d \rangle$ be a metric space and let $h\colon X \to R$. If f is a continuous function that majorizes h, define a sequence of continuous functions $\{f_k\}$ as follows:* (i) $f_1 = f - \varphi(f, h)$, (ii) *for each* $k > 1$, $f_k = f_{k-1} - \varphi(f_{k-1}, h)$. *Then $\{f_k\}$ is a decreasing sequence of functions convergent pointwise to h^*, the upper envelope of h.*

Proof. By Lemma 1 for each x in X and each $k \in Z^+$, we have $h^*(x) \leqslant f_{k+1}(x) \leqslant f_k(x)$. Suppose for some x, $\beta = \inf_k f_k(x)$ exceeds $h^*(x)$. Since h^* is u.s.c. there exists $\lambda > 0$ such that $\lambda < 1/2(\beta - h^*(x))$, and whenever $d(w, x) < \lambda$ then $h^*(w) < 1/2(\beta + h^*(x))$. Choose k so large that $f_k(x) - \beta < \lambda$. It follows that

$$\varphi(f_k, h)(x) = f_k(x) - f_{k+1}(x) < \lambda.$$

Hence there is a point w such that both $d(w, x) < \lambda$ and $h(w) > f_k(x) - \lambda$. But then $h^*(w) \geqslant h(w) > f_k(x) - \lambda > 1/2(\beta + h^*(x))$, and this contradicts the choice of λ.

If our function h is u.s.c., i.e., $h = h^*$, then the algorithm described in the statement of Theorem 1 produces a decreasing sequence of continuous functions convergent pointwise to h, and if h is majorized by a Lipschitz function, then by Lemma 1, we can generate such a sequence of Lipschitz functions. If X is compact, h will have a Lipschitz majorant, for h will attain a maximum value. Otherwise, no such majorant need exist. If X is unbounded, fix w and for each $n \in Z^+$ choose x_n such that $d(w, x_n) > n$. Then $h\colon X \to R$ defined by

$$h(x) = d(w, x_n)^2, \qquad \text{if } x = x_n \text{ for some } n,$$
$$= 0, \qquad \text{otherwise,}$$

is u.s.c. and has no Lipschitz majorant. If X is bounded but noncompact, let $\{x_n\}$ be a sequence in X with no convergent subsequence. Define $h: X \to R$ by

$$h(x) = n, \quad \text{if } x = x_n \text{ for some } n,$$
$$= 0, \quad \text{otherwise.}$$

Again h is u.s.c. with no Lipschitz majorant. Still, it is always easy to initiate the algorithm. Let $\psi: R \to (-1, 1)$ be a bicontinuous increasing bijection, e.g., $\psi(x) = x/(1+|x|)$. Since ψ is increasing, if h is u.s.c., then $\psi \circ h$ will be u.s.c. Let g map each point of X to 1. Since $\psi \circ h$ is u.s.c. and for each x, $(\psi \circ h)(x) < g(x)$, by the remark following Lemma 1, we have for all x

$$g(x) - \varphi(g, \psi \circ h)(x) < 1.$$

Thus $f = \psi^{-1} \circ (g - \varphi(g, \psi \circ h))$ is a continuous majorant of h and can be used to initiate the algorithm.

2. Rate of Convergence

Let X be an arbitrary metric space, $h: X \to R$ an upper semicontinuous function, and $f: X \to R$ a continuous function that majorizes h. Under what circumstances will the algorithm described in the previous section produce a sequence of continuous functions that converge uniformly to h? Continuity of h is an obvious necessary condition, but it is far from sufficient. Actually, uniform continuity of h is necessary in the following sense: if $h \in C(X, R)$ is not uniformly continuous, then there exists $f \in C(X, R)$ majorizing h for which $\{f_k\}$ defined by (i) $f_1 = f - \varphi(f, h)$, (ii) $f_k = f_{k-1} - \varphi(f_{k-1}, h)$, for $k > 1$, fails to converge uniformly to f; moreover, if h is bounded, we can choose f to be bounded. We first obtain a lemma that puts a limit on the rate of growth of $|f_k(x) - f(x)|$.

Lemma 2. *Let $f: X \to R$ be continuous and let $h: X \to R$ be arbitrary with $f \geqslant h$. Let $f_1 = f - \varphi(f, h)$, and for each $k > 1$, let $f_k = f_{k-1} - \varphi(f_{k-1}, h)$. Then for each $k \in Z^+$ and for each x in X,*

$$f_k(x) \geqslant f(x) + (1 - 2^k)\, \varphi(f, h)(x).$$

Proof. For $k = 1$, we actually have equality. Assume the inequality holds for $k = j$ and set $\delta = \varphi(f, h)$. By the definition of δ for each $\varepsilon > 0$, there exists $y \in X$ for which $\rho[(x, f(x)), (y, h(y))] < \delta + \varepsilon$, whence

$$f_j(x) - \varepsilon \leqslant f(x) - \delta - \varepsilon < h(y) < f(x) + \delta + \varepsilon.$$

By the induction hypothesis

$$f_j(x) - \varepsilon < h(y) \leqslant f_j(x) + 2^j\delta + \varepsilon,$$

and since $d(x, y) < 2^j\delta$, we have

$$\varphi(f_j, h)(x) \leqslant 2^j\delta + \varepsilon.$$

As a result,

$$\begin{aligned} f_{j+1}(x) &= f_j(x) - \varphi(f_j, h)(x) \\ &\geqslant [f(x) + (1 - 2^j)\delta] - 2^j\delta \\ &= f(x) + (1 - 2^{j+1})\delta. \end{aligned}$$

We also need an interposition theorem of Michael [4].

MICHAEL'S THEOREM. *Let X be a topological space in which each closed set is a G_δ set. Suppose h: $X \to R$ is* u.s.c., *g: $X \to R$ is* l.s.c., *and $g \geqslant h$. Then there exists $f \in C(X, R)$ that* ultra-strictly interposes [1] *h and g: $h \leqslant f \leqslant g$, and whenever $h(x) < g(x)$, we have $h(x) < f(x) < g(x)$.*

This result of course applies if X is metric. In this context suppose $h \in C(X, R)$ is not uniformly continuous. For some $\varepsilon > 0$ and for each $k \in Z^+$ there exists points x_k and y_k in X such that $d(x_k, y_k) < 2^{-(k+1)}\varepsilon$ and $h(y_k) > h(x_k) + \varepsilon$. Let ψ: $R \to (-1, 1)$ be a bicontinuous increasing bijection. Now $\{x_k : k \in Z^+\}$ can have no limit points, or else h would not be continuous. Thus h^* $X \to (-1, 1)$ defined by

$$\begin{aligned} h^*(x) &= \psi(h(y_k)), \qquad &&\text{if} \quad x = x_k, \\ &= \psi(h(x)), &&\text{otherwise,} \end{aligned}$$

is u.s.c. Define g^*: $X \to (-1, 1]$ by

$$\begin{aligned} g^*(x) &= \psi(h(y_k)), \qquad &&\text{if} \quad x = x_k, \\ &= 1, &&\text{otherwise.} \end{aligned}$$

Since $\{x_k : k \in Z^+\}$ is closed, g^* is l.s.c. By Michael's theorem there exists $f^* \in C(X, (-1, 1])$ which ultra-strictly interposes h^* and g^*; note that actually $f^* \in C(X, (-1, 1))$ so that $f = \psi^{-1} \circ f^* \in C(X, R)$. Clearly, $f \geqslant h$ and $f(x_k) = h(y_k)$. Now for each $k \in Z^+$,

$$\rho[(x_k, f(x_k)), (y_k, h(y_k))] < 2^{-(k+1)}\varepsilon,$$

whence $\varphi(f, h)(x_k) < 2^{-(k+1)}\varepsilon$. By Lemma 2,

$$\begin{aligned} f_k(x_k) &\geqslant f(x_k) - 2^k \varphi(f, h)(x_k) \\ &\geqslant h(y_k) - 2^k \cdot 2^{-k-1}\varepsilon \\ &> h(x_k) + \varepsilon/2. \end{aligned}$$

We conclude that $\{f_k\}$ does not converge uniformly to h.

Our next goal is to show that if h is a bounded uniformly continuous function then uniform convergence occurs. We need the following simple fact.

LEMMA 3. *Let $h: X \to R$ be arbitrary and let $f \in C(X, R)$ majorize h. Let $f_1 = f - \varphi(f, h)$, and for each $k > 1$, let $f_k = f_{k-1} - \varphi(f_{k-1}, h)$. Then for each $k \in Z^+$, $\sup_{x \in X} f_k(x) = \sup_{x \in X} h(x)$.*

Proof. It suffices to show this is true for $k = 1$. Fix x in X and let $\varepsilon > 0$ be arbitrary. Suppose $\varphi(f, h)(x) = \delta$. Choose $y \in X$ for which $\rho[(x, f(x)), (y, h(y))] < \delta + \varepsilon$. In particular $f(x) < h(y) + \delta + \varepsilon$ whence $f_1(x) < h(y) + \varepsilon$. This proves $\sup_{x \in X} f_1(x) \leqslant \sup_{x \in X} h(x)$; the reverse inequality follows from Lemma 1.

THEOREM 2. *Let $h: X \to R$ be a bounded uniformly continuous function on a metric space X, and let $f \in C(X, R)$ majorize h. Then if $f_1 = f - \varphi(f, h)$ and for each $k > 1$ $f_k = f_{k-1} - \varphi(f_{k-1}, h)$, then $\{f_k\}$ converges uniformly to h.*

Proof. Let $M = \sup_{x \in X} |h(x)|$. Suppose the convergence is not uniform. Then for some $\varepsilon > 0$ there exists a sequence $\{x_n\}$ in X and a subsequence $\{f_{k_n}\}$ of $\{f_k\}$ such that for each n, $f_{k_n}(x_n) > h(x_n) + \varepsilon$. Choose $\lambda > 0$ such that whenever $d(x, y) < \lambda$ then $|h(x) - h(y)| < \varepsilon/2$, and set $\theta = \min\{\lambda, \varepsilon/2\}$. We claim that whenever $j \leqslant k_n$ that $\varphi(f_j, h)(x_n) \geqslant \theta$. If not, there exists $y \in X$ such that $\rho[(x_n, f_j(x_n)), (y, h(y))] < \theta$. It follows that $d(x_n, y) < \lambda$ and

$$h(y) > f_j(x_n) - \frac{\varepsilon}{2} \geqslant f_{k_n}(x_n) - \frac{\varepsilon}{2} > h(x_n) + \frac{\varepsilon}{2}$$

in violation of the choice of λ. Now choose n so large that $n\theta > 2M$. By Lemma 3, $\sup_{x \in X} f_1(x) \leqslant M$; so,

$$\begin{aligned} M \geqslant f_1(x_n) &\geqslant \sum_{j=1}^{k_n} \varphi(f_j, h)(x_n) + h(x_n) \\ &\geqslant \sum_{j=1}^{n} \varphi(f_j, h)(x_n) + h(x_n) \\ &> 2M + h(x_n). \end{aligned}$$

We have shown $h(x_n) < -M$, a contradiction.

Theorem 2 fails without further assumptions if h is allowed to be unbounded.

EXAMPLE 1. Let X be the following metric subspace of the line:

$$X=\{n^2: n\in Z^+ \text{ and } n\geqslant 2\}\cup\{n^2+1: n\in Z^+ \text{ and } n\geqslant 2\}.$$

Since distinct points in X have distance at least one from one another, each real function on X is uniformly continuous. Define $h: X\to R$ by

$$\begin{aligned} h(x)&=0, && \text{if } x=n^2 \text{ for some } n,\\ &=n, && \text{if } x=n^2+1 \text{ for some } n. \end{aligned}$$

Let $f: X\to R$ map both n^2 and n^2+1 to n, $n=2, 3, 4,\ldots$. Note that for each n, $\varphi(f, h)(n)=1$. Let $k\in Z^+$ be arbitrary and choose n so large that $n/2>2^k$. By Lemma 2,

$$\begin{aligned} f_k(n)&\geqslant n+(1-2^k)(1)\\ &>n-\frac{n}{2}>h(n)+\frac{1}{2}. \end{aligned}$$

Thus $\{f_k\}$ does not converge uniformly to h.

A generally weaker requirement than uniform convergence of $\{f_k\}$ is uniform convergence of $\{\varphi(f_k, h)\}$. Intuitively uniform convergence of the latter sequence means that eventually the points of the graph of f_k are uniformly close to the graph of h, but not necessarily measured vertically. If X is relatively nice (as described below) and h is a bounded continuous function, then $\{\varphi(f_k, h)\}$ will converge uniformly.

DEFINITION. A metric space $\langle X, d\rangle$ is *radially connected* if for each $(a, b)\in X\times X$ there exists a connected set $K(a, b)$ containing both a and b such that for each $w\in K(a, b)$, $d(a, w)\leqslant d(a, b)$.

Evidently convex sets in normed linear spaces are radially connected. More generally, $\langle X, d\rangle$ is called *convex* if for each a and b in X there exists m in X such that $d(a, m)=d(b, m)=(1/2)\, d(a, b)$ [7]. If, in addition, X is complete, then for each a and b in X there exists a path φ from a to b such that for each $\tau\in[0, 1]$, $d(a, \varphi(\tau))=\tau d(a, b)$ and $d(b, \varphi(\tau))=(1-\tau)\, d(a, b)$ [2]. Thus, such spaces are radially connected. But there are other examples: a circle in the plane is radially connected.

LEMMA 4. *Let X be a radially connected metric space, and let $h: X\to R$ and $f: X\to R$ be continuous with $f\geqslant h$. Then for each $k\in Z^+$, we have $\varphi(f_{k-1}, h)(x)\geqslant\varphi(f_k, h)(x)$.*

Proof. We first claim that if g is continuous and $g \geqslant h$ and $\varphi(g, h)(x) = \delta$, then for each $\varepsilon > 0$ there exists $y \in X$ such that

(1) $d(x, y) \leqslant \delta + \varepsilon$,

(2) $g(x) - \delta - \varepsilon \leqslant h(y) \leqslant g(x) - \delta$.

This is clearly true if $\delta = 0$, for then $g(x) = h(x)$, and we can choose $y = x$. Also, if $h(x) = g(x) - \delta$, then we can also choose $y = x$. Otherwise, since $h(x) \leqslant g(x)$ and $\delta \leqslant |g(x) - h(x)|$, we must have $h(x) < g(x) - \delta$. Without loss of generality we can assume that $\varepsilon < g(x) - \delta - h(x)$. Now pick $(z, h(z))$ for which $\rho[(x, g(x)), (z, h(z))] < \delta + \varepsilon$. We have

$$h(x) < g(x) - \delta - \varepsilon < h(z) < g(x) + \delta + \varepsilon.$$

Let $K(x, z)$ be the connected subset of X containing x and z such that for each $w \in K(x, z)$, $d(x, w) \leqslant d(x, z)$. Since $h(K(x, z))$ is connected, $\exists y \in K(x, z)$ such that $h(y) = g(x) - \delta - \varepsilon$. Since $d(x, y) \leqslant d(x, z) < \delta + \varepsilon$, this choice of y works, and the claim is established.

Suppose now that f_{k-1} has been defined. Of course $f_k(x) = f_{k-1}(x) - \varphi(f_{k-1}, h)(x)$. By the above argument with $g = f_{k-1}$ for each $\varepsilon > 0$, there exists $y \in X$ for which

(1) $d(x, y) \leqslant \varphi(f_{k-1}, h)(x) + \varepsilon$,

(2) $f_k(x) - \varepsilon \leqslant h(y) \leqslant f_k(x)$.

It follows that $\rho[(x, f_k(x)), (y, h(y))] \leqslant \varphi(f_{k-1}, h)(x) + \varepsilon$, whence $\varphi(f_k, h)(x) \leqslant \varphi(f_{k-1}, h)(x)$.

We remark that if X is actually a closed subset of R^n, then one can show that strict inequality occurs; i.e., whenever $\varphi(f_{k-1}, h)(x) \neq 0$, we have $\varphi(f_k, h)(x) < \varphi(f_{k-1}, h)(x)$.

THEOREM 3. *Let X be a radially connected metric space. Let $h\colon X \to R$ be a bounded continuous function and let $f\colon X \to R$ be a continuous function that majorizes h. Let $f_1 = f - \varphi(f, h)$, and for each $k > 1$, let $f_k = f_{k-1} - \varphi(f_{k-1}, h)$. Then $\{\varphi(f_k, h)\}$ converges uniformly to the zero function.*

Proof. If the convergence is not uniform for some $\varepsilon > 0$ and for each $N \in Z^+$, there exists $k > N$ and $x \in X$ for which $\varphi(f_k, h)(x) > \varepsilon$. Since

$$f_1(x) \geqslant \sum_{j=1}^{k} \varphi(f_j, h)(x) + h(x),$$

Lemma 4 says that $f_1(x) \geqslant k\varepsilon + h(x)$. Since N was arbitrary and h is bounded, f_1 cannot be bounded above, contradicting Lemma 3.

Theorem 3 fails if X is merely connected instead of radially connected.

EXAMPLE 2. For $n = 3, 4, 5, \ldots$ let A_n and B_n be the following subsets of the plane:

$$A_n = \{(n, y) : 0 \leqslant y \leqslant 2\},$$

$$B_n = \left\{\left(n + \frac{1}{n}, y\right) : 0 \leqslant y \leqslant 2\right\}.$$

Set $A = \bigcup_{n=3}^{\infty} A_n$ and $B = \bigcup_{n=3}^{\infty} B_n$ and let $C = \{(x, 2) : x \geqslant 3\}$. Then $X = A \cup B \cup C$ as a metric subspace of the plane is polygonally connected, but is not radially connected. Define $h \in C(X, [-1, 0])$ by

$$\begin{aligned} h(x, y) &= -1, && \text{if } (x, y) \in C \cup A, \\ &= 0, && \text{if } (x, y) \in B \text{ and } y \leqslant 1, \\ &= -y + 1, && \text{if } (x, y) \in B \text{ and } 1 < y \leqslant 2. \end{aligned}$$

Let $f: X \to R$ be the zero function. For each $n \in \{3, 4, 5, \ldots\}$, a nearest point to $(n, 0, 0) = (n, 0, f(n, 0))$ on the graph of h is $(n + 1/n, 0, 0)$; this remains true for $(n, 0, f_k(n, 0))$ as long as $f_k(n, 0) \geqslant -1/2$. Clearly, for all such k, $f_k(n, 0) = -2^{k-1}(1/n)$. Hence if k is arbitrary and we set $n = 2^k$, then $f_k(n, 0) = -1/2$ so that $\varphi(f_k, h)(n, 0) = 1/2$. Thus $\{\varphi(f_k, h)\}$ does not converge uniformly to zero.

Theorem 3 also fails without further assumptions if h is allowed to be unbounded.

EXAMPLE 3. We present an unbounded continuous function h with domain R^+ and a Lipschitz function f majorizing h for which $\{\varphi(f_k, h)\}$ fails to converge uniformly. We describe h by describing its graph: it is the infinite polygonal path in R^2 joining the following points in succession: $(0, 0)$, $(8/3, -3)$, $(3, 0)$, $(10/3, -3)$, $(15/4, -4)$, $(4, 0)$, $(17/4, -4)$, $(24/25, -5)$, $(5, 0)$, $(26/25, -5), \ldots$. Clearly, h is majorized by the zero function f. For $n = 3, 4, 5, \ldots$, we have $\varphi(f, h)(n + 1/4) \leqslant 1/4$. Let $k \in Z^+$ be arbitrary and choose $n \geqslant 8$ so large that $2^{k \; 2} \leqslant n$. By Lemma 2, we have

$$f_k\left(n + \frac{1}{4}\right) \geqslant (1 - 2^k)\left(\frac{1}{4}\right) \geqslant -n + \frac{1}{4}.$$

Now if $d(x, n + 1/4) \leqslant 1/8$, we have $h(x) \leqslant -n$. Thus for each $x \in X$, we obtain $\rho[(n + 1/4, f_k(n + 1/4)), (x, h(x))] \geqslant 1/8$, whence $\varphi(f_k, h)(n + 1/4) \geqslant 1/8$.

For uniformly continuous h, uniform convergence of $\{f_k\}$ is equivalent to uniform convergence of $\{\varphi(f_k, h)\}$. One direction is immediate, for

$$f_k(x) - h(x) = \sum_{j=k}^{\infty} \varphi(f_j, h)(x) \geqslant \varphi(f_k, h)(x).$$

On the other hand, suppose $\{\varphi(f_k, h)\}$ converges uniformly to zero. Let $\varepsilon > 0$ be arbitrary and choose $\delta < \varepsilon/2$ such that whenever $d(w, z) < \delta$, then $|h(w) - h(z)| < \varepsilon/2$. Choose N so large that for each $k > N$ and for all x, $\varphi(f_k, h)(x) < \delta$. Fix $x \in X$ and $k > N$ and choose w in X such that $\rho[(x, f_k(x)), (w, h(w))] < \delta$. Since $d(x, w) < \delta$, we have $|h(w) - h(x)| < \varepsilon/2$, and since $\delta < \varepsilon/2$, it follows that $|f_k(x) - h(x)| < \varepsilon$. It should be noted that the same reasoning can be used to prove the following fact: if X and Y are arbitrary metric spaces, then the Hausdorff metric as applied to graphs of uniformly continuous functions from X to Y gives the topology of uniform convergence. This equivalence is the basis for a number of papers in constructive approximation theory (see, e.g., [5, 8, 9, 11]).

References

1. J. Blatter and G. Seever, Interposition of semicontinuous functions by continuous functions, *in* "Analyse Fonctionelle et Applications (Comptes rendus du Colloque d'Analyse, Rio de Janeiro, 1972)," pp. 27–51, Hermann, Paris, 1975.
2. L. Blumenthal, "Theory and Applications of Distance Geometry," Oxford Univ. Press (Clarendon), 1953.
3. H. Hahn, Über halbstetige und unstetige funktionen, *Sitzungsber. Akad. Wiss. Wien Abt. IIa* **126** (1917), 91–110.
4. E. Michael, Continuous selections, I, *Ann. of Math.* **63** (1956), 361–382.
5. V. Popov, Approximation of convex functions by algebraic polynomials in Hausdorff metric, *Serdica* **1** (1975), 386–398.
6. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N.J., 1970.
7. D. Rolfsen, A metric characterization of the 3-cell, *Fund. Math.* **68** (1970), 215–223.
8. Bl. Sendov, Several questions on the theory of approximation of functions and sets in the Hausdorff metric, *Uspekhi Mat. Nauk.* **24** (1969), 141–178.
9. Bl. Sendov, and B. Penkov, Hausdorffsche metrik und approximation, *Numer. Math.* **9** (1966), 214–226.
10. H. Tong, Some characterizations of normal and perfectly normal spaces, *Duke Math. J.* **19** (1952), 289–292.
11. V. Veselinov, Approximation of unbounded functions with linear positive operators in Hausdorff metric, *C. R. Bulg. Acad. Sci.* **22** (1969), 499–502.